UDC 62-50

POINT EVASION CONDITIONS IN A SECOND-ORDER DIFFERENTIAL GAME

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We derive the necessary and sufficient conditions for the evasion of a point in a nonlinear second-order differential game. These conditions are defined concretely for the case of a linear differential game. The paper is related to [1–6].

1. We consider the second-order system

$$dx/dt = F(x, u, v), \quad u \in U, \quad v \in V \tag{1.1}$$

Here x is a phase vector, u (v) is the first (second) player's control. The function $F(x, u, v)$ is continuous in all its arguments and satisfies a Lipschitz condition in x, U and V are closed bounded sets. We assume that for any x and any $v \in V$ the set $F(x, U, v) = U_u F(x, u, v)$, $u \in U$, is convex. By the termination of the game we mean the hitting of system (1.1) into a certain preassigned point m. We assume that a vector p exists for which the scalar product $p_1F_1(m, u, v) + p_2F_2(m, u, v) < 0$ for all $u \in U$, $v \in V$.

Let "the realization $u(\cdot)$" be an arbitrary measurable function $u(t)$, $t_0 \leqslant t < \infty$ satisfying the condition $u(t) \in U$ for any t. We shall assume that when $t \geqslant t_0$ the second player can collide with any realization $u(\cdot)$. He should choose his own control by means of the descrete scheme $\{v[x], \Delta\}$. The discretum $\Delta > 0$ defines the size of the semi-interval $t^* \leqslant t < t^* + \Delta$ during which the control $v = v[x(t^*)]$ is held constant. By $T[x_0; v[x], \Delta, u(\cdot)]$ the transition time of system (1.1) to point m from an initial position $x_0 = x(t_0)$ under the discrete scheme $\{v[x], \Delta\}$ and the realization $u(\cdot)$. If such a transition is not possible, we set $T[x_0; v[x], \Delta, u(\cdot)] = \infty$.

D e f i n i t i o n. An evasion is possible in the game if there exist functions $v^\circ[x]$, $\Delta[x_0]$ such that for any $x_0 \neq m$ and for all $\Delta \leqslant \Delta[x_0]$, $u(\cdot)$ the time $T[x_0; v[x], \Delta, u(\cdot)] = \infty$.

2. Without loss of generality we assume that the origin of the rectangular system of coordinates x_1, x_2 coincides with m, and that the vector p is directed along the x_1-axis. Let O be a closed circle with center at m, for any point x of which we have

$$F_1(x, u, v) < 0 \tag{2.1}$$

for all $u \in U$, $v \in V$. We set

$$f(x, u, v) = \frac{F_2(x, u, v)}{F_1(x, u, v)}$$

$$f^*(x) = \max_v \min_u f(x, u, v) \tag{2.2}$$

$$f_*(x) = \min_v \max_u f(x, u, v) \tag{2.3}$$

$$x \in O, \quad u \in U, \quad v \in V$$

From the definition of the functions $f^*(x)$, $f_*(x)$ and from the fulfillment of a

Lipschitz condition in x for the function $F(x, u, v)$ it follows that the functions $f^*(x), f_*(x)$ satisfy in O a Lipschitz condition in x. Let $x_2 = \psi^*(x_1)$ ($x_2 = \psi_*(x_1)$), $x_1 \geqslant 0$ be a solution of the equation $dx_2/dx_1 = f^*(x)$ ($dx_2/dx_1 = f_*(x)$) with the initial condition $x_2(0) = 0$, continued up to the boundary of circle O. We denote the graph of this solution by r^* (r_*). We say that $r^* > r_*$ if at the intersection of the positive semiaxis of x_1 with circle O we can find a monotonically decreasing sequence of points $\{x_1^{(n)}\}$ converging to zero, for which $\psi^*(x_1^{(n)}) > \psi_*(x_1^{(n)})$ for any n. Otherwise we say that $r^* \leqslant r_*$.

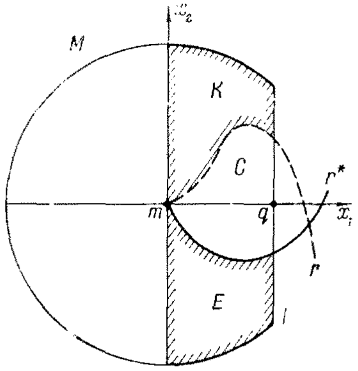

Fig. 1

Theorem 2.1. For evasion to be possible it is necessary and sufficient that the condition $r^* > r_*$ be satisfied.

The proof follows from Lemmas 2.1, 2.2. Let

$$f^{(1)}(x, v) = \min_u f(x, u, v), \quad f^{(2)}(x, v) = \max_u f(x, u, v)$$
$$x \in O, \quad u \in U, \quad v \in V$$

Lemma 2.1. If $r^* \leqslant r_*$, evasion is impossible.

Proof. Since $r^* \leqslant r_*$, we can find a number $q > 0$ such that $\psi^*(x_1) \leqslant \psi_*(x_1)$ for $x_1 \in [0, q]$. Let O° be the interior of circle O. We set (see Fig. 1)

$$M = O^\circ \cap \{x : x_1 < q\}, \qquad H = M \cap \{x : x_1 \geqslant 0\}$$
$$C = \{x : x \in H, \quad \psi^*(x_1) \leqslant x_2 \leqslant \psi_*(x_1)\}$$
$$K = \{x : x \in H, \quad x_2 > \psi_*(x_1)\}$$
$$E = \{x : x \in H, \quad x_2 < \psi^*(x_1)\}$$

For any $x \in M$, $v \in V$ we assume

$$U^{(i)}(x, v) = \{u : u \in U, \quad f(x, u, v) = f^{(i)}(x, v)\}, \quad i = 1,2$$

$$U(x, v) = \begin{cases} U^{(1)}(x, v) & \text{if} \quad x \in E \\ U^{(2)}(x, v), & \text{if} \quad x \in K \\ U, & \text{if} \quad x \in M \setminus (K \cup E) \end{cases}$$

For every $v \in V$ the set $U(x, v)$ is upper semicontinuous relative to inclusion with respect to x in M (see [7], Theorem 1.1). Consequently, the set $F(x, v) = U_u F(x, u, v)$, $u \in U(x, v)$, possesses an analogous property. From the convexity of set $F(x, U, v)$ for all $x \in M$, $v \in V$ follows the convexity of set $F(x, v)$ for all $x \in M$, $v \in V$.

Let us consider the differential equation $dx/dt \in F(x, v)$. For any initial condition $x_0 \in M$ and for any constant $v \in V$ it has at least one solution [8]. Since $f^{(1)}(x, v) \leqslant f^*(x)$ ($f^{(2)}(x, v) \geqslant f_*(x, v)$) in E (K), any solution $x(t)$ starting at an arbitrary point $x_0 = x(t_0) \in C$ does not go, for $t \geqslant t_0$, above (below) the curve r_* (r^*) till the first instant t^* of reaching the boundary of set H. Consequently, the solution stays in C and hits point m in the time $t^* - t_0 < \vartheta = q / j$, where $j = \min | F_1(x, u, v) |$ on the product $O \times U \times V$. As follows from the lemma (in [9], p. 27], from the solution $x(t)$ we can select a measurable function $u(t), t \in [t_0, t^*]$,

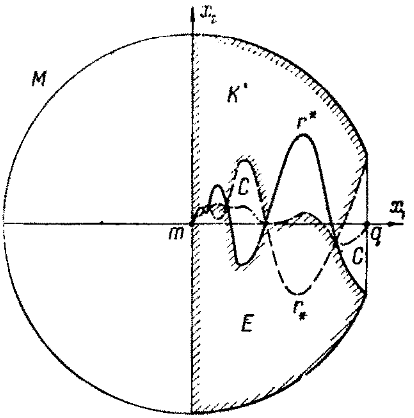

Fig. 2

with values in the set $U(x(t), v)$, such that the solution of the equation $dx/dt = F(x, u(t), v)$ (x_0 and v are as before) coincides with $x(t)$ on the interval $[t_0, t^*]$. From what we have said it follows that for any $x_0 \in C$ and for any descrete scheme $\{v[x], \Delta\}$ we can find a realization $u(\cdot)$ for which the time $T[x_0; v[x], \Delta, u(\cdot)] < \vartheta$. Evasion is impossible. The proof is completed.

By $V^*(x)$ $(V_*(x))$, $x \in O$, we denote the maximal collection of vectors $v \in V$ on each of which the maximum (minimum) is reached in (2.2) ((2.3)).

Lemma 2.2. If $r^* > r_*$, evasion is possible.

Proof. Let $[0, q]$ be the largest common segment of definition of the functions $\psi^*(x_1)$ and $\psi_*(x_1)$. We set

$$\psi(x_1) = {}^1/_2(\psi^*(x_1) + \psi_*(x_1)), \quad s(x_1) = \psi^*(x_1) - \psi_*(x_1)$$
$$x_1 \in [0, q]$$
$$\omega^{(1)}(x) = \psi^*(x_1) - x_2, \qquad \omega^{(2)}(x) = x_2 - \psi_*(x_1)$$
$$x \in \{x : x_1 \in [0, q]\}$$
$$M = O^\circ \cap \{x : x_1 < q\}, \qquad H = M \cap \{x : x_1 > 0\}$$
$$C = \{x : x \in H, \quad \psi^*(x_1) \leqslant x_2 \leqslant \psi_*(x_1)\}$$
$$D = \{x_1 : x_1 \in [0, q], \quad \psi^*(x_1) > \psi_*(x_1)\}$$
$$K = \{x : x \in H \setminus C, \quad x_2 \geqslant \psi(x_1)\}$$
$$E = \{x : x \in H \setminus C, \quad x_2 < \psi(x_1)\}$$

The notation introduced is clarified in Fig. 2. The solid (dotted) line shows the curve r^* (r_*), the dashed-dotted line shows the curve $x_2 = \psi(x_1)$, $x_1 \in [0, q]$. We define

$$v^\circ[x] = \begin{cases} \text{any} \quad v \in V_*(x), & \text{if} \quad x \in K \\ \text{any} \quad v \in V^*(x), & \text{if} \quad x \in E \\ \text{any} \quad v \in V, & \text{if} \quad x \overline{\in} K \cup E \end{cases}$$

1) Let $x_0 = x(t_0) \in (M \setminus \{m\}) \setminus H$ and let the second player apply the discrete scheme $\{v^\circ[x], \Delta\}$. As a consequence of (2.1) and of the fact that on the product $O \times U \times V$ the function $|f(x, u, v)|$ is bounded from above by the number $G = \max |f(x, u, v)|$, we obtain that for all Δ, $u(\cdot)$ the motion $x(t)$ of system (1.1), from the instant t_0 up to the first instant t^* of reaching the boundary of set M, goes in the sector $\{x : G(x_1 - x_{10}) + x_{20} \leqslant x_2 \leqslant -G(x_1 - x_{10}) + x_{20},$ $x_1 \leqslant x_{10}\}$, and hence

$$|x(t)| \geqslant \frac{|x_0|}{\sqrt{1 + G^2}}, \qquad t \in [t_0, t^*] \tag{2.4}$$

2) Let $y = x(t_*) \in E$ and let a constant $v = v^\circ[y]$ and an arbitrary realization

$u(\cdot)$ act on the semi-interval $[t_*, t_* + \Delta)$, where Δ is fairly small. Let us estimate the function $\omega^{(1)}(x(t))$, $t \in [t_*, t_* + \Delta)$. Since $f^*(y) = f^{(1)}(y, v^\circ[y])$ and the functions $f^*(x)$, $f^{(1)}(x, v)$ satisfy a Lipschitz condition in x (with a common constant k), we have

$$\begin{gathered} f^*(x(t)) - f(x(t),\ u(t),\ v^\circ[y]) \leqslant \\ f^*(x(t)) - f^{(1)}(x(t),\ \ v^\circ[y]) \leqslant \\ |f^*(x(t)) - f^*(y)| + |f^{(1)}(y, v^\circ[y]) - \\ f^{(1)}(x(t),\ \ v^\circ[y])| \leqslant 2k|x(t) - y| \leqslant \\ 2kN\Delta, \quad t \in [t_*, t_* + \Delta), \end{gathered} \tag{2.5}$$

where $N = \max |F(x, u, v)|$ on $O \times U \times V$. From (2.1), (2.5) it follows that on the semi-interval $[t_*, t_* + \Delta)$ the motion $x(t)$ with any $u(\cdot)$ does not go above the integral curve of the equation $\partial x_2/\partial x_1 = f^*(x) - 2kN\Delta$, drawn through point y. Since the curve r^* is an integral curve of the equation $\partial x_2/\partial x_1 = f^*(x)$, issuing from the point m, then

$$\begin{gathered} \omega^{(1)}(x(t)) \geqslant (\omega^{(1)}(y) + 2N\Delta)\exp(-k|x_1(t) - y_1|) - 2N\Delta \\ t \in [t_*, t_* + \Delta) \end{gathered} \tag{2.6}$$

Analogously, if $y = x(t_*) \in K$ and if a constant $v = v^\circ[y]$ and an arbitrary realization $u(\cdot)$ act on the semi-interval $[t_*, t_* + \Delta)$, where Δ is fairly small, then

$$\begin{gathered} \omega^{(2)}(x(t)) \geqslant (\omega^{(2)}(y) + 2N\Delta)\exp(-k|x_1(t) - y_1|) - 2N\Delta \\ t \in [t_*, t_* + \Delta) \end{gathered} \tag{2.7}$$

Let $x_0 \in H$. We set

$$\begin{gathered} \chi(x_0) = \max_{x_1}(s(x_1)\exp(-kx_1)) \\ x_1 \in [0, x_{10}] \cap D \end{gathered} \tag{2.8}$$

The smallest x_1 at which the maximum in (2.8) is reached, we denote by $\alpha(x_0)$. We fix an initial position $x_0 = x(t_0) \in H$ and assume that the second player applies the discrete scheme $\{v^\circ[x], \Delta\}$, where $\Delta \leqslant \Delta[x_0]$ and $\Delta[x_0]$ is fairly small.

Suppose that up to the first instant of intersection with the straight line $x_1 = \alpha(x_0)$ the motion $x(t)$ from point x_0 takes place in H and that the indicated intersection occurs at some l th semi-interval of the discrete scheme (l is a positive integer depending on Δ and $u(\cdot)$). From the definition of function $v^\circ[x]$ and from inequalities (2.6), (2.7), we obtain that for a fairly small $\Delta[x_0]$ the motion $x(t)$ for all $\Delta \leqslant \Delta[x_0]$, $u(\cdot)$, from the instant $t_1 = t_0 + \Delta(l - 1)$ up to the first instant t_2 of leaving H, goes at each discretum Δ either strictly above the curve r_* or strictly below the curve r^*, and

$$\begin{gathered} \max\{\omega^{(1)}(x(t)),\ \omega^{(2)}(x(t))\} \geqslant \chi(x_0) - \xi(\Delta) \\ t \in [t_1, t_2] \end{gathered} \tag{2.9}$$

Here and below $\xi(\Delta)$ denotes a positive first-order infinitesimal as $\Delta \to 0$. Since the functions $|f^*(x)|$, $|f_*(x)|$ do not exceed the number G, the maximum $\lambda(t)$ of the distances from point $x(t)$ up to the curves r^*, r_* at the instant t is estimated by the inequality

$$\begin{gathered} \lambda(t) \geqslant \frac{\chi(x_0)}{\sqrt{1 + G^2}} - \xi(\Delta) \overset{\text{def}}{=} \mu(x_0, \Delta) \\ t \in [t_1, t_2] \end{gathered} \tag{2.10}$$

From (2.4), (2.9), (2.10) it follows that for any $t \geqslant t_1$ we have $|x(t)| \geqslant \mu(x_0, \Delta)$. up to the first instant t^* of departure from M. Obviously, on the interval $[t_0, t^*]$ we have $|x(t)| \geqslant \min\{\mu(x_0, \Delta), \alpha(x_0)\}$. Since $\alpha(x_0) \geqslant s(\alpha(x_0))/G > \mu(x_0, \Delta)$, then

$$|x(t)| > \mu(x_0, \Delta), \qquad t \in [t_0, t^*] \tag{2.11}$$

Thus, if $x_0 = x(t_0) \in H$ and if the second player applies the discrete scheme $\{v^\circ[x], \Delta\}$, then for fairly small $\Delta[x_0]$ the motion of system (1.1) for all $\Delta \leqslant \Delta[x_0], u(\cdot)$ cannot approach the point m, on the interval $[t_0, t^*]$, where t^* is the first instant of departure from M, closer than at the distance $\mu(x_0, \Delta) = \chi(x_0)/\sqrt{1+G^2} - \xi(\Delta)$.

3) For $x_1 \in D$ we set

$$\nu^{(i)}(x_1) = \min_y (\omega^{(i)}(y)\exp(-ky_1))$$
$$y \in \{y : y \overline{\in} M, y_1 \in [0, x_1], \omega^{(i)}(y) > 0\}, \ i = 1, 2$$
$$\nu(x_1) = s(x_1)\exp(-kx_1), \qquad \varkappa = \max_{x_1}\min\{\nu^{(1)}(x_1), \nu^{(2)}(x_1), \nu(x_1)\}$$

Using the results of items (1), (2) we can show that if $x_0 = x(t_0) \overline{\in} M$ and if the second player applies the discrete scheme $\{v^\circ[x], \Delta\}$, then for all $\Delta \leqslant \delta, u(\cdot)$, where $\delta > 0$ is fairly small and independent of x_0,

$$|x(t)| \geqslant \frac{\varkappa}{\sqrt{1+G^2}} - \xi(\Delta), \quad t \geqslant t_0 \tag{2.12}$$

From (2.4), (2.11), (2.12) we obtain that for any $x_0 \neq m$ we can find $\Delta[x_0]$ such that for all $\Delta \leqslant \Delta[x_0]$, $u(\cdot)$ the time $T[x_0; v^\circ[x], \Delta, u(\cdot)] = \infty$. The proof is completed.

It is evident that $r^* > r_*$ $(r^* \leqslant r_*)$ when $f^*(m) > f_*(m)$ $(f^*(m) < f_*(m))$, therefore, from Theorem 2.1 follows –

Corollary 2.1. If $f^*(m) > f_*(m)$ $(f^*(m) < f_*(m))$, then evasion is possible (impossible).

3. Let us consider the two-dimensional system

$$dx/dt = Ax + u - v, \qquad u \in U, \qquad v \in V \tag{3.1}$$

Here A is a constant 2×2 matrix, U and V are closed bounded convex sets. We assume that $U \cap V = \phi$ and that at least one of the sets U or V is a polygon. We take the origin as the point m. Theorem 2.1 is valid for system (3.1) under the stated assumptions. Below we indicate simpler necessary and sufficient evasion conditions for system (3.1) than the conditions in Theorem 2.1. We select the coordinate system and the circle O in the same way as in Sect. 2. We denote the straight line $x_2 = f^*(m)x_1$ by β, and its part, when $x_1 > 0$ by α.

Theorem 3.1. For evasion to be possible it is necessary and sufficient that at least one of the following two conditions be fulfilled: (1) $f^*(m) > f_*(m)$, (2) $f^*(m) = f_*(m)$ and there exists a circle $L \subset O$ with center at point m, such that $f^*(x) > f_*(x)$ for any $x \in \alpha \cap L$.

An analogous theorem was stated in [6] in somewhat different terms and under stricter assumptions. The theorem's proof is based on Lemma 3.1 which is considered below. Assume that $f^*(m) = f_*(m)$ and let the straight line β not be invariant relative to the transformation A corresponding to matrix A. Then the set $\gamma = \{x : Ax \in \beta\}$ is a straight line passing through point m and not coinciding with β. That one of the halfplanes defined by the straight line γ, which contains the the half-line α, is called Γ. We do not include the straight line γ in the halfplane Γ. Let $C(l) = \{x : x \in O,$

$|x| < l\}$, $l > 0$. If $f^*(m) = f_*(m)$ and the straight line β is invariant (is not invariant), we set $D(l) = C(l) \cap \beta$ $\quad (D(l) = C(l) \cap \Gamma)$.

Lemma 3.1. Let $f^*(m) = f_*(m)$. Then there exists a number $l^\circ > 0$ such that either $f^*(x) > f_*(x)$ for any $x \in D(l^\circ)$ or $f^*(x) \leqslant f_*(x)$ for any $x \in D(l^\circ)$.

Proof. If the straight line β is invariant, the lemma is obvious. Suppose that the straight line β is not invariant. Since $f^*(m) = f_*(m)$, for all $v^* \in V^*(m)$ and $v_* \in V_*(m)$ the sets $-U + v^*$ and $-U + v_*$ are separated (but not strictly) by the straight line β. We set

$$P(v) = (-U + v) \cap \beta$$

$$\rho^* = \max_v \min \{|w| : w \in P(v)\}, \quad v \in V^*(m) \tag{3.2}$$

$$\rho_* = \min_v \max \{|w| : w \in P(v)\}, \quad v \in V_*(m) \tag{3.3}$$

Three cases are possible: (1) $\rho^* > \rho_*$, (2) $\rho^* < \rho_*$, (3) $\rho^* = \rho_*$. The transformation A maps the halfplane Γ into one of the halfplanes defined by the straight line β. We denote it by B. The straight line β does not occur in B. Below the analysis of cases (1)–(3) is carried out under the assumption that the halfplane B lies above the straight line β (see Fig. 3). The arguments are analogous if it lies below.

We set

$$\varphi(z, u, v) = \frac{z_2 + u_2 - v_2}{z_1 + u_1 - v_1}$$

$$\varphi^*(z) = \max_v \min_u \varphi(z, u, v) \tag{3.4}$$

$$\varphi_*(z) = \min_v \max_u \varphi(z, u, v) \tag{3.5}$$

$$z \in A(O), \quad u \in U, \quad v \in V$$

By h^* (h_*) we denote the vector $v \in V^*(m)$ $(v \in V_*(m))$, on which the maximum (minimum) is reached in (3.2) ((3.3)). In case (1), $P(h^*) \cap P(h_*) = \varnothing$. Therefore, $(-U + h^*) \cap (-U + h_*) = \varnothing$. Consequently, we can find a sufficiently small $l_0 > 0$ such that for any $z \in K(l_0) = C(l_0) \cap B$

$$\varphi^*(z) \geqslant \min_u \varphi(z, u, h^*) > \max_u \varphi(z, u, h_*) \geqslant \varphi_*(z)$$
$$u \in U$$

By virtue of the continuity of transformation A and of the equalities $f^*(x) = \varphi^*(Ax)$, $f_*(x) = \varphi_*(Ax)$ there follows the existence of a sufficiently small $l^\circ > 0$ such that for any $x \in D(l^\circ)$ we have $f^*(x) > f_*(x)$.

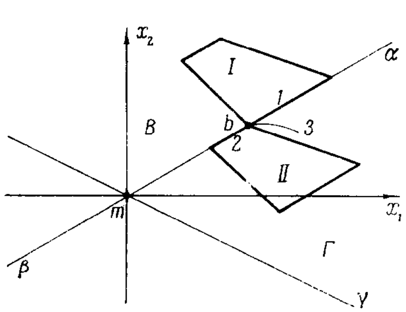

Fig. 3

In case (2) we show the existence of $l_0 > 0$ such that for any $z \in K(l_0)$

$$\varphi^*(z) < \varphi_*(z) \tag{3.6}$$

We assume the contrary. Then we can isolate a sequence $\{z^{(n)}\}$ of points from $B_{(n)}$, converging to m, for which $\varphi^*(z^{(n)}) \geqslant \varphi_*(z^{(n)})$ for any n. With each point $z^{(n)}$ we associate the pair $(v^{(n)}, v_n)$, where $v^{(n)}$ (v_n) is an arbitrary vector from V on which the maximum (minimum) is reached in (3.4) ((3.5)) for $z = z^{(n)}$ From the sequence $\{v^{(n)}, v_n\}$ we select a convergent subsequence $\{v^{(k)}, v_k\}$. Let (v°, v_0) be its limit. It is obvious that $v^\circ \in V^*(m)$, $v_0 \in V_*(m)$. Since the segment $E^{(k)} = P(h^*) + v^{(k)} - h^* \subset -U + v^{(k)}$ lies in the sector

$$\{x : \varphi^*(z^{(k)})(x_1 - z_1^{(k)}) + z_2^{(k)} \leqslant x_2 \leqslant f^*(m)\, x_1\}$$

while the segment $E_k = P(h_*) + v_k - h_* \subset -U + v_k$ lies in the sector

$$\{x : f^*(m)\, x_1 \leqslant x_2 \leqslant \varphi_*(z^{(k)})(x_1 - z_1^{(k)}) + z_2^{(k)}\}$$

From the conditions
$$\varphi^*(z^{(k)}) \geqslant \varphi_*(z^{(k)}), \qquad k = 1, 2, \ldots$$
$$P(v^\circ) = \lim_{k\to\infty} E^{(k)}, \qquad P(v_0) = \lim_{k\to\infty} E_k$$

we obtain that

$$\min\{|w| : w \in P(v^\circ)\} \geqslant \max\{|w| : w \in P(v_0)\}$$

The latter contradicts the condition $\rho^* < \rho_*$. Thus, (3.6) is valid. From (3.6) follows the existence of $l^\circ > 0$ such that $f^*(x) < f_*(x)$ for any $x \in D(l^\circ)$.

In case (3) the intersection $P(h^*) \cap P(h_*)$ consists of one point which we denote by the letter b. We assume at first that tne set U is a polygon. There are two possibilities: (3a) the boundary of set V is tangent at the point h^* to the straight line $x_2 - h_2^* = f^*(m)\ (x_1 - h_1^*)$, from the right, or else it is tangent at the point h_* to the straight line $x_2 - h_{2*} = f^*(m)(x_1 - h_{1*})$; from the left; (3b) condition (3a) is not satisfied.

In case (3a) we assume, for definiteness, that the tangency is from the right at the point h^*. Let χ be an arc of the boundary of set V, abutting h^* from the right. It is not difficult to see that if the arc χ is fairly small, then for any $v \in \chi$ the set $-U + v$ (U is a polygon) lies above the arc $\chi + b - h^*$. Consequently, for a fairly small $l_0 > 0$

$$\varphi^*(z) > \frac{z_2 - b_2}{z_1 - b_1} \geqslant \varphi_*(z) \tag{3.7}$$

for any $z \in K(l_0)$. Case (3a) is shown in Fig. 3. The numeral I (II) denotes the set $-U + h^*$ $(-U + h_*)$, the numeral 1 (2) denotes the segment $P(h^*)$ $(P(h_*))$. The arc $\chi + b - h^*$ is denoted by numeral 3.

In case (3a), for a fairly small $l^\circ > 0$,

$$\varphi^*(z) = \frac{z_2 - b_2}{z_1 - b_1} = \varphi_*(z) \tag{3.8}$$

for any $z \in K(l_0)$. From (3.7) ((3.8)) it follows that in case (3a) ((3b)) there exists $l_0 > 0$ such that $f^*(x) > f_*(x)$ $(f^*(x) = f_*(x))$ for any $x \in D(l^\circ)$. If set V is a polygon, then for fairly small $l_0 > 0$

$$\varphi^*(z) \leqslant \frac{z_2 - b_2}{z_1 - b_1} \leqslant \varphi_*(z)$$

and, hence, there exists $l^\circ > 0$ such that $f^*(x) \leqslant f_*(x)$ for any $x \in D(l^\circ)$. The proof is completed.

We return to Theorem 3.1. Let condition (2) of Theorem 3.1 be satisfied (let $f^*(m) = f_*(m)$, but let condition (2) not be satisfied). Then, by Lemma 3.1 there exists $l^\circ > 0$ such that $f^*(x) > f_*(x)$ $(f^*(x) \leqslant f_*(x))$ for any $x \in D(l^\circ)$. From the geometry of set $D(l^\circ)$ and from the definition of curves r^*, r_* it follows that in this case $r^* > r_*$ $(r^* \leqslant r_*)$. Theorem 3.1 now follows from Theorem 2.1 and Corollary 2.1.

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ON AN ESTIMATE IN A DIFFERENTIAL GAME OF ENCOUNTER

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We consider the game problem of the encounter of a conflict-controlled phase point with a specified target set M. We give an upper bound of the result achieved by feedback control in nonregular cases. The construction is based on the ideas in [1, 2].

1. We consider a controlled system described by the differential equation

$$x^{\cdot} = A(t)\,x + B(t)\,u - C(t)\,v \tag{1.1}$$

Here x is the system's n-dimensional phase vector; $A(t)$, $B(t)$ and $C(t)$ are continuous matrices; u and v are the r-dimensional vectors of the controlling forces at the disposal of the first and second players, respectively. The realizations $u[t]$, $v[t]$ of controls u, v are constrained by the conditions

$$u[t] \in P, \qquad v[t] \in Q \tag{1.2}$$

where P and Q are closed, bounded, and convex sets. We examine the conflict problem of the encounter of the point $x[t]$ with a specified closed convex set M: the first player's aim is the encounter, the second player's aim is to prevent it. The problem is considered on a fixed time interval $[t_0, \vartheta]$. As the game's cost we choose the quantity

$$\gamma = \rho(x[\vartheta], M) \tag{1.3}$$

where the symbol $\rho(x, M)$ denotes the distance from point x to set M. We shall adhere to the definitions presented in [1] for the player's strategy classes and for the